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# Two Level Correction Algorithm for Parametric Shape Inverse Optimization

Jichao Zhao<sup>1,2,\*</sup>, Badr Abou El Majd<sup>2,3</sup>, Jean Antoine Désidéri<sup>2</sup>

<sup>1</sup> Auckland Bioengineering Institute, University of Auckland, New Zealand.

<sup>2</sup> Opale Project, INRIA Sophia Antipolis, France.

<sup>3</sup> LIAD Lab., Hassan II University, Morocco.

Email: j.zhao@auckland.ac.nz

\*Corresponding author.

## ABSTRACT

Our efforts are mostly concentrated on improving the convergence rate of the numerical procedures both from the viewpoint of cost-efficiency and accuracy by handling the parametrization of the shape to be optimized. We employ nested parameterization supports of either shape, or shape deformation, and the classical process of degree elevation resulting in exact geometrical data transfer from coarse to fine representations. The algorithms mimic classical multigrid strategies and are found very effective in terms of convergence acceleration. In this paper, we analyse and demonstrate the efficiency of the two-level correction algorithm which is the basic block of a more general multilevel strategy.

## KEYWORDS

Parametric shape optimization — RAE2822 airfoil — Model problem — Multilevel algorithms — Optimum shape-design in aerodynamics.

## 1. Introduction

Our developments focus on the definition, mathematical analysis and experimentation of numerical methods for shape optimization for applications in which the cost functional evaluation relies on the prior solution of a complex set of partial-differential equations (PDEs), such as those governing compressible aerodynamics (e.g. the Euler equations), or related coupled disciplines such as structural mechanics (e.g. elasticity), or electromagnetics (e.g. the Maxwell equations). These PDEs are very commonly solved by Finite Element or Volume Methods, by techniques that, although becoming increasingly standard, are still very costly when the accuracy requirement is high. The important development of multigrid methods in recent years has demonstrated that such techniques not only permit to accelerate the iterative convergence of solution procedures, but also have the more general merit of a better control on grid dependency and convergence.

This contribution is a sequel of [1]-[5] in which we are mostly concentrated on improving the convergence rate of numerical procedures both from the viewpoint of cost-efficiency and accuracy, with the perspective of reducing the design cost, but also of mastering the election and control of the design parameters, geometrical ones in particular, in a more rational way, perhaps supported by error estimates.

We present in this paper a model problem for shape optimization for purpose of the analysis of the two-level correction algorithm which is the basic block of a more general multilevel strategy. We first consider the shape reconstruction problem based on Bézier parametrization. Then, we analyse and show the efficiency of three two-level correction algorithms for a linear and a nonlinear case. We finally conclude with some perspectives.

## 2. Coarse-level correction algorithms for a linear problem

We use an intrinsic formulation of shape-reconstruction or shape-inverse problem, initially introduced in [6]:

$$\min_{\gamma} \mathcal{J}(\gamma) := \int_{\gamma} \frac{1}{2} [y(x) - \bar{y}(x)]^2 dx \quad (1)$$

where  $\gamma$  is the unknown shape analytically represented by  $y(x)$ ;  $\bar{y}(x)$  is the analogous analytical representation of a given target curve  $\bar{\gamma}(x)$ , subsequently assumed, without great loss of generality, to be a Bézier curve of degree  $n$  and support  $X$ . This problem is transformed into a parametric optimization by assuming Bézier representation of the curves over the support  $X$ :

$$\min_{Y \in \mathbf{R}^{n+1}} j_n(Y) := \int_0^1 \frac{1}{2} [B_n(t)^T (Y - \bar{Y})]^2 n B_{n-1}(t)^T \Delta X dt, \quad (2)$$

where  $B_n(t)^T = (B_n^0(t), B_n^1(t), \dots, B_n^n(t))$  is the  $1 \times (n+1)$  vector of Bernstein polynomials and the symbol  $\Delta$  represents the forward-difference operator that appears when differentiating Bernstein polynomials. For the uniform case, we know that  $x^0(t)' = n B_{n-1}(t)^T \Delta X^0 = 1$ , so we only need to care about

$$j_n(Y) = \int_0^1 \frac{1}{2} (B_n(t)^T (Y - \bar{Y}))^2 dt. \quad (3)$$

Since the functional is quadratic, the parametric gradient is linear (in  $Y$ ):

$$j'_n(Y) = AY - b \quad (4)$$

where

$$A = \int_0^1 B_n(t) B_n(t)^T n B_{n-1}(t)^T \Delta X dt \quad (5)$$

and

$$b = \int_0^1 B_n(t) B_n(t)^T \bar{Y} n B_{n-1}(t)^T \Delta X dt \quad (6)$$

In particular, for a uniform support  $X$ , the matrix  $A$  reduces to the simple form :

$$A = \int_0^1 B_n(t) B_n(t)^T dt = A_{ij} \quad (7)$$

in which the coefficients  $A_{ij}$  are obtained by a simple calculation :

$$A_{ij} = \frac{1}{2n+1} \frac{C_n^i C_n^j}{C_{2n}^{i+j}}, \quad (8)$$

and the right side known vector

$$b = A\bar{Y}. \quad (9)$$

For this shape-inverse problem, the optimization problem (2) is equivalent to solving the linear system:

$$AY = b \quad (10)$$

With the usual multigrid terminology, we define a two-level correction algorithm in two steps as follow:

1. In the fine level, we use the classical steepest-descent iteration as relaxation method:

$$Y^{j+1} = Y^j - \rho (AY^j - b), \quad (11)$$

For  $K$  sweeps of descent method ( $1 \leq j \leq K$ ), we obtain an approximate optimum shape corresponding to the design vector  $Y^K$  of Bézier parametrization of degree  $n$ .

2. In the coarse level, the unknown is defined to be a correction vector to be prolonged, here by the degree elevation operator  $E_{n'}^n$ , and added to the freshest update of the unknown vector coming from the fine level iteration, here  $Y^K$ . We note that  $n'(n' < n)$  is the dimension of the coarse level space and  $E_{n'}^n$  is the matrix associated with the linear process of  $n - n'$  degree elevations.

## 2.1 Coarse-level correction: $Y$ method

The first approach consists to solve completely the optimization problem:

$$\min_{Y' \in \mathbf{R}^{n'+1}} J(Y') := j_n(E_{n'}^n Y' + Y_k) \quad (12)$$

Let  $Y = E_{n'}^n Y' + Y_k$ , it's obvious to prove that

$$\frac{\partial J(Y')}{\partial Y'} = \frac{\partial Y}{\partial Y'} \frac{\partial J(Y')}{\partial Y}, \quad (13)$$

$$\frac{\partial Y}{\partial Y'} = (E_{n'}^n)^T, \quad (14)$$

and

$$\frac{\partial J(Y')}{\partial Y} = A(E_{n'}^n Y' + Y_k - \bar{Y}). \quad (15)$$

Thus by equations (13), (14) and (15), we obtain

$$\begin{aligned} \frac{\partial J(Y')}{\partial Y'} &= \frac{\partial Y}{\partial Y'} \frac{\partial J(Y')}{\partial Y} \\ &= (E_{n'}^n)^T A(E_{n'}^n Y' + Y_k - \bar{Y}) = 0. \end{aligned} \quad (16)$$

In other words, the problem (12) is equivalent to the linear system:

$$A_{cy} Y' = b_{cy}, \quad (17)$$

here the coefficient matrix

$$A_{cy} = (E_{n'}^n)^T A E_{n'}^n, \quad (18)$$

and the right side vector

$$b_{cy} = (E_{n'}^n)^T A(-Y_k + \bar{Y}) = (E_{n'}^n)^T (b - A Y_k). \quad (19)$$

We can solve the following iteration on the coarse level for correction by using:

$$Y'^{j+1} = Y'^j - \rho(A_{cy} Y'^j - b_{cy}) \quad (Y'_0 = 0), \quad (20)$$

then we can update by  $Y^K + E_{n'}^n Y'$  on the fine level. Iterations on the fine level (11) and corrections on the coarse level (20) complete a two-level correction-type ideal algorithm for the linear model problem. However, it takes many iterations (hundreds for this model problem) to achieve a complete convergence. To speed up the rate of convergence, we can use a Tchebychev iterations [7] on the fine level, i.e., it has three steps for each cycle:

$$\begin{aligned} Y^{j1} &= Y^{j0} - \tau_1 (A Y^{j0} - b), \\ Y^{j2} &= Y^{j1} - \tau_2 (A Y^{j1} - b), \\ Y^{j3} &= Y^{j2} - \tau_3 (A Y^{j2} - b), \end{aligned}$$

here  $\tau_i$  ( $i = 1, 2, 3$ ) are multiplicative inverses of the three roots of the half frequent (HF) Tchebychev polynomial for the matrix  $A$ . When we can solve coarse corrections exactly on the coarse level, we can combine the Tchebychev iterations (21) with coarse corrections on the coarse level in the matrix form [7]:

$$G_y = G_h (I - E_{n'}^n ((E_{n'}^n)^T A E_{n'}^n)^{-1} (E_{n'}^n)^T A) G_h, \quad (21)$$

here  $G_h = (I - \tau_3 A)(I - \tau_2 A)(I - \tau_1 A)$ . In other works, the complete cycle in the form of matrix  $G$  is given by

$$Y_g^{j+1} = G_y Y_g^j + b_{gy}, \quad (22)$$

here

$$b_{gy} = G_h (b_h - E_{n'}^n ((E_{n'}^n)^T A E_{n'}^n)^{-1} (E_{n'}^n)^T (A b_h - b)) + b_h, \quad (23)$$

and

$$b_h = ((I - \tau_3 A)(I - \tau_2 A)\tau_1 + (I - \tau_3 A)\tau_2 + \tau_3 I)b. \quad (24)$$

## 2.2 Coarse level correction: Z Method

The matrix  $A$  is real-symmetric definite positive and can thus be diagonalized by an orthogonal transformation:

$$A = \Omega_n \Lambda_n \Omega_n^T, \quad (25)$$

The diagonal matrix  $\Lambda_n$  has real positive eigenvalues, arranged in increasing order, and the column-vectors of the orthogonal matrix  $\Omega_n$  are the associated eigenvectors

$$\Omega_n \Omega_n^T = \Omega_n^T \Omega_n = I. \quad (26)$$

By using this decomposition of the matrix  $A$ , the second approach use the following scheme:

$$\min_{Z' \in \mathbf{R}^{n'+1}} J(Z') := j_n(Y_k + Q_0 E_{n'}^n Z') \quad (27)$$

where  $Q_0 = \Omega_n P_n \Omega_n^T$  and  $P_n$  is the  $(n+1) \times (n+1)$  matrix associated with the permutation:

$$\{0, 1, \dots, n-1, n\} \longrightarrow \{n, n-1, \dots, 1, 0\} \quad (28)$$

The idea of the Z method is to reverse pairing between eigenvalues and eigenvectors by multiplying by the matrix  $Q_0$ . Hence, the classical modal-analysis concept of multigrid can be applied straightforwardly. See [8][9] for more details about the compatibility of the cooarse level iteration with the eigensystem analysis.

Let  $Z = (Q_0 E_{n'}^n) Z' + Y_k$ , it's easy to prove that

$$\frac{\partial J(Z')}{\partial Z'} = \frac{\partial Z}{\partial Z'} \frac{\partial J(Z)}{\partial Z}, \quad (29)$$

$$\frac{\partial Z}{\partial Z'} = (Q_0 E_{n'}^n)^T = (E_{n'}^n)^T Q_0, \quad (30)$$

and

$$\frac{\partial J(Z')}{\partial Z} = A(Y_k + Q_0 E_{n'}^n Z' - \bar{Y}). \quad (31)$$

Thus by equations (29), (30) and (31), we obtain

$$\begin{aligned} \frac{\partial J(Z')}{\partial Z'} &= \frac{\partial Z}{\partial Z'} \frac{\partial J(Z)}{\partial Z} \\ &= (E_{n'}^n)^T Q_0 A(Y_k + Q_0 E_{n'}^n Z' - \bar{Y}) = 0. \end{aligned} \quad (32)$$

In other words, the second approach is equivalent to the linear system:

$$A_{cz} Z' = b_{cz}, \quad (33)$$

here the matrix

$$A_{cz} = (E_{n'}^n)^T Q_0 A Q_0 E_{n'}^n = (E_{n'}^n)^T A_1 E_{n'}^n, \quad (34)$$

and the vector

$$b_{cz} = (E_{n'}^n)^T Q_0 (b - A Y_k), \quad (35)$$

Equivalently, we can solve the following iteration on the coarse level for correction by initializing  $Z'_0 = 0$ :

$$Z'^{j+1} = Z'^j - \rho (A_{cz} Z'^j - b_{cz}) \quad (Z'_0 = 0), \quad (36)$$

then, we can update the obtained vector design by  $Y^K + Q_0 E_{n'}^n Z'$  and speed up the convergence rate by using Tchebychev iterations.

### 2.3 Coarse Level Correction: $L$ Method

$L$  or  $\Delta$  Method uses the idea of the  $Z$  Method to reverse pairing between eigenvalues and eigenvectors by a simple matrix  $\Delta$  instead of computing the complicated matrix  $Q_0$ . The matrix  $\Delta$  is given by:

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & (-1)^{n-1} & 0 \\ 0 & \cdots & 0 & 0 & (-1)^n \end{pmatrix}_{(n+1) \times (n+1)},$$

Let  $L = (\Delta E_{n'}^n)L' + Y_k$ . As the previous section, we can prove easily that:

$$A_{cl}L' = b_{cl}, \quad (37)$$

here the matrix

$$A_{cl} = (E_{n'}^n)^T \Delta A \Delta E_{n'}^n = (E_{n'}^n)^T \Delta A \Delta E_{n'}^n, \quad (38)$$

and the vector

$$b_{cl} = (E_{n'}^n)^T \Delta (b - AY_k), \quad (39)$$

This system can be solved by the following iteration on the coarse level for correction by initializing  $L'_0 = 0$ :

$$L'^{j+1} = L'^j - \rho(A_{cl}L'^j - b_{cl}), \quad (40)$$

then, we can update by  $Y^K + \Delta E_{n'}^n L'$  on the fine level and we can use Tchebychev iterations [7] to speed up the rate of convergence.

### 2.4 Numerical experiments

We take,

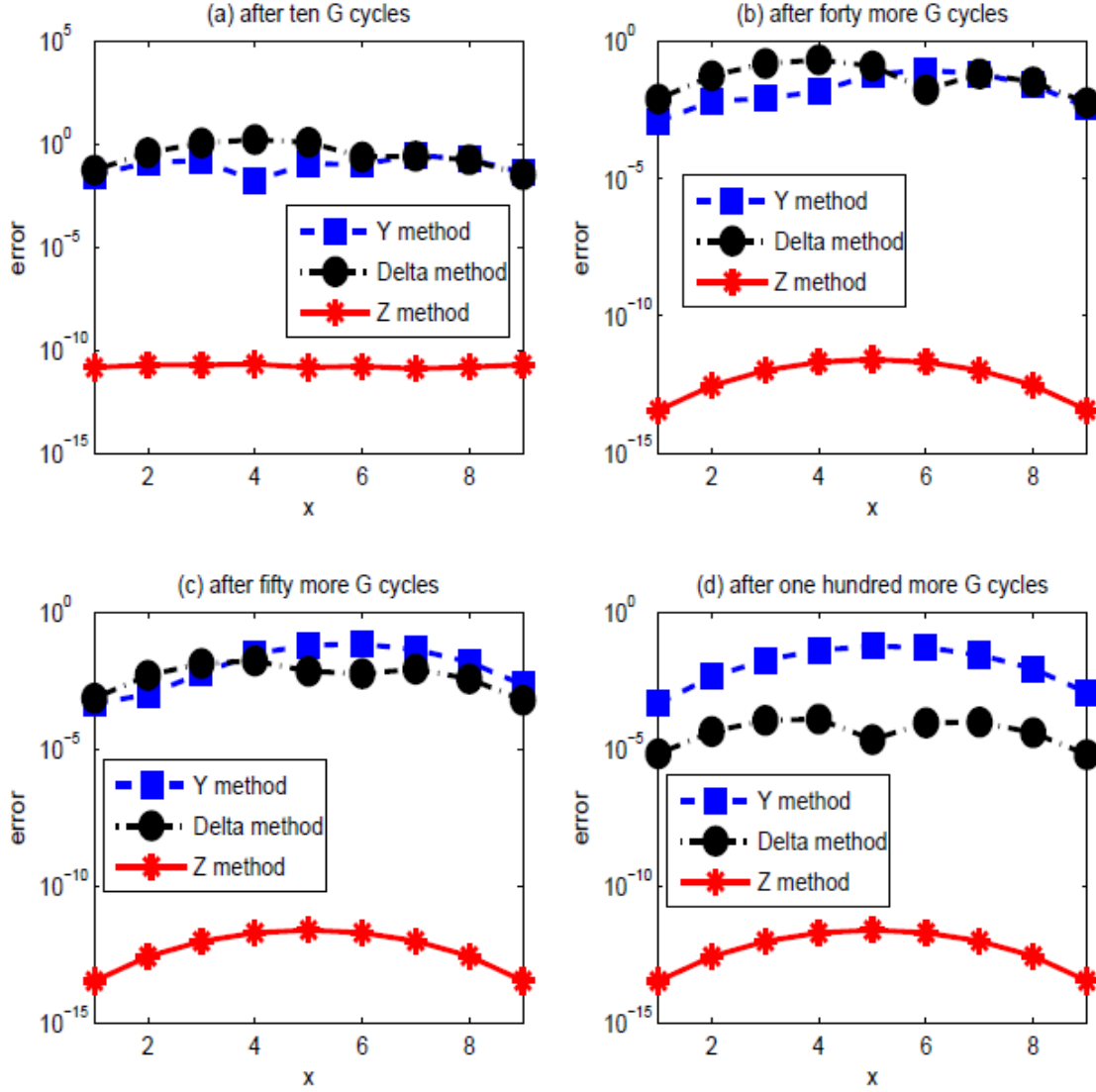
$$\bar{Y}(x) = \sum_{k=1}^n \frac{1}{k^2} \sin(k\pi x) \quad (41)$$

so that the solutions can have all the frequencies, which makes computational results and our analysis more generally. When we analyse our numerical experiments, the error is defined as the difference between the approximation  $Y^j$  obtained by different methods and the true solution  $\bar{Y}$ . We use Tchebychev iterations on the fine level to speed up the convergence rate and solve analytically on the coarse level, then we plot errors versus the number of iterations in Fig. 1, and frequency versus the number of iterations in Fig. 2. In Fig. 1, we still randomly generate initial errors, (a), (b), (c) and (d) show errors obtained by  $Y$ ,  $L$  and  $Z$  methods after ten, fifty, one hundred and two hundred  $G$  cycles respectively. It is obvious that the results obtained by  $Z$  method is the best. And from these figures, we can see that before the first 50 iterations, results obtained by  $L$  method doesn't look better than those obtained by  $Y$  method, but in (c) and (d), it shows that  $L$  method is better than  $Y$  method. From the corresponding frequency space (Fig. 2 (c) and (d)), we can see that  $L$  method also can reduce low frequency on the fine level, however it takes more iterations to converge, the reason is that  $L$  method introduces some errors by the transformation matrix  $\Delta$  and it needs more iterations on the fine level to remove some errors introduced by the coarse corrections.

## 3. Nonlinear Model Problem

In this section, we extend the concept of coarse level coorection for the nonlinear problem. We consider the nonlinear model problem or the inverse shape problem:

$$\min \mathcal{J} = \mathcal{J}(y(t)) = \frac{p^\alpha}{\mathcal{A}}, \quad (42)$$



**Figure 1.** Nodal components. (a), (b), (c) and (d) show errors obtained by Y, L and Z methods after ten, fifty, one hundred and two hundred G cycles respectively.

in which  $x(t)$  is given, smooth and monotone-increasing,

$$p = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} \omega(t) dt, \quad \mathcal{A} = \int_0^1 x'(t)y(t)\omega(t) dt, \quad (43)$$

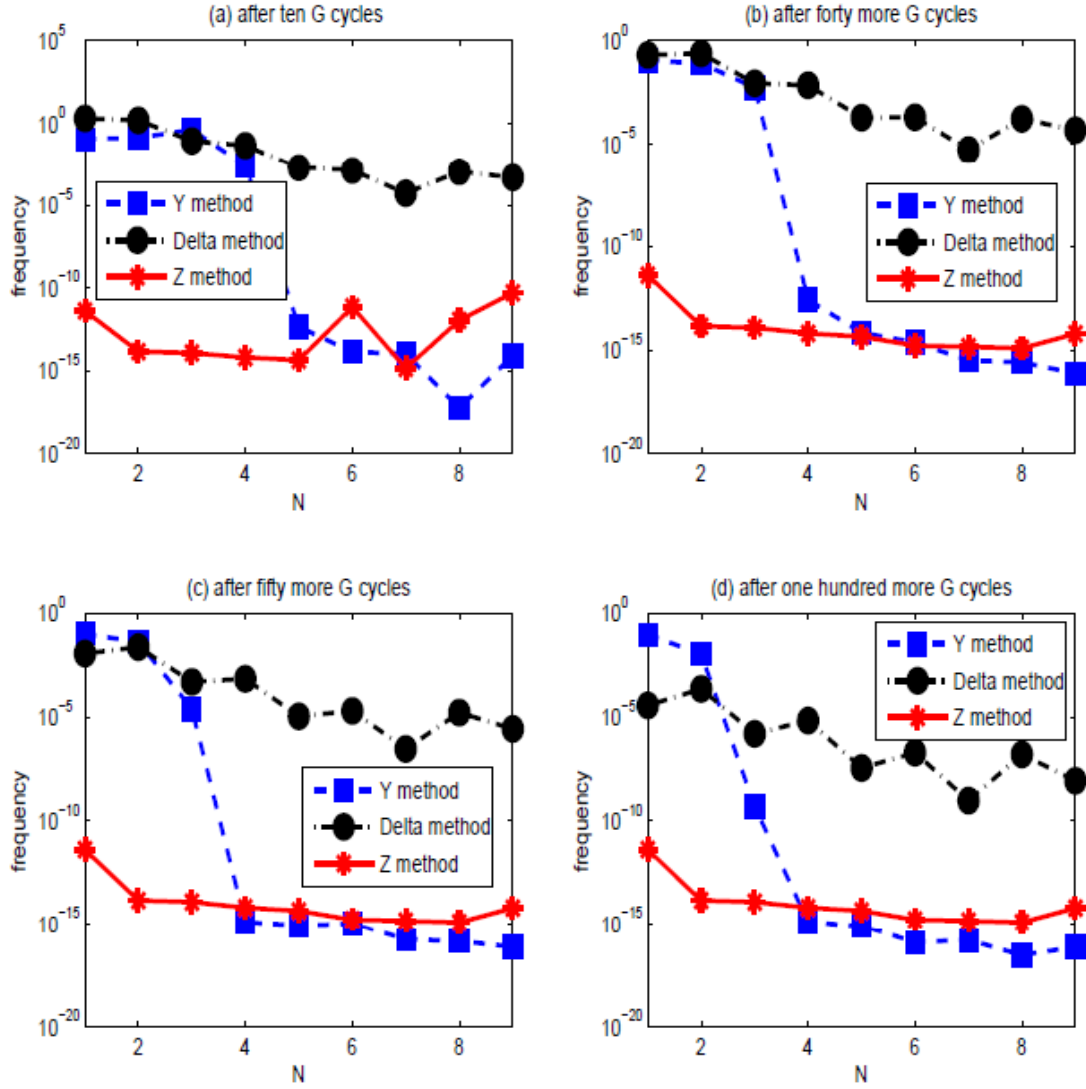
are, for specified  $\omega(t) > 0$  and  $\alpha > 1$ , the pseudo-length of the arc, and the pseudo-area below the arc. Essentially all smooth unimodal graphs can be retrieved by this formulation [10]. In the following experiments, algorithms are tested for  $\alpha = 2$  and  $\omega(t) = 1$  for each  $t$ . In this case, the minimum value for this problem is  $\mathcal{J} = 2\pi$ .

For this nonlinear problem, we define a two-level correction algorithm in two steps as follow:

1. In the fine level, we use the classical steepest-decent iteration as relaxation method:

$$Y^{j+1} = Y^j - \rho \mathcal{J}'(Y^j), \quad (44)$$

where  $\mathcal{J}'(Y^j)$  is the Jacobian matrix denoted by  $A_{\mathcal{J}}$ .



**Figure 2.** Frequency components. (a), (b), (c) and (d) show frequency obtained by Y, L and Z methods after ten, fifty, one hundred and two hundred G cycles respectively.

For  $K$  sweeps of descent method ( $1 \leq j \leq k$ ), we obtain an approximate optimum shape corresponding to the design vector  $Y^K$  of Bézier parametrization of degree  $n$ .

2. In the coarse level, the unknown is defined to be a correction vector to be prolonged, here by the degree elevation operator  $E_n^n$ , and added to the freshest update of the unknown vector coming from the fine level iteration, here  $Y^k$ .

In the following, we discuss a various algorithms of coarse level corrections.

### 3.1 Coarse Level Correction: Y Method

For the Y method, we set

$$Y = Y^K + E_n^n Y', \quad (45)$$

here  $Y^K$  is the value obtained on the fine level, and  $Y'$  is the coarse level correction we are looking for.

By using coarse level correction (45) in the original nonlinear problem (42), we can easily obtain the following



relations for the  $Y'$  correction:

$$\mathcal{J}'(Y') = E_{n'}^{nT} \cdot \mathcal{J}'(Y) = A'_{\mathcal{J}'} Y' - b'_{\mathcal{J}'}, \quad (46)$$

here  $A'_{\mathcal{J}'}$  stands for the coarse level Jacobian matrix, its value can be obtained by

$$A'_{\mathcal{J}'} = E_{n'}^{nT} A_{\mathcal{J}'} E_{n'}^n, \quad (47)$$

and the constant vector  $b'_{\mathcal{J}'}$

$$b'_{\mathcal{J}'} = E_{n'}^{nT} (b_{\mathcal{J}'} - A_{\mathcal{J}'} Y^K), \quad (48)$$

$A_{\mathcal{J}'}$  is the Jacobian matrix on the fine level. Thus we can again employ classical steepest-descent iterations on the coarse level for correction by initializing  $Y'_0 = 0$ :

$$Y' \leftarrow Y' - \rho \mathcal{J}'(Y'), \quad (49)$$

then we can update by  $Y^K + E_{n'}^n Y'$  on the fine level. Iterations on the fine level (44) and corrections on the coarse level (49) complete a two-level correction-type idea algorithm for the nonlinear model problem. However it takes many iterations (hundreds for this model problem) to achieve complete convergence. To speed up the rate of convergence, we can use a better technique – Tchebychev iterations [7] on the fine level, i.e., it has three steps for each cycle:

$$\begin{aligned} Y^{j_1} &= Y^{j_0} - \tau_1 \mathcal{J}'(Y^{j_0}), \\ Y^{j_2} &= Y^{j_1} - \tau_2 \mathcal{J}'(Y^{j_1}), \\ Y^{j_3} &= Y^{j_2} - \tau_3 \mathcal{J}'(Y^{j_2}), \end{aligned}$$

here  $\tau_i$  ( $i = 1, 2, 3$ ) are multiplicative inverses of the three roots of the half frequent (HF) Tchebychev polynomial for the matrix  $A_{\mathcal{J}'}$ . Note that the matrix  $A_{\mathcal{J}'}$  is corresponding to  $Y^{j_0}$ , so we can only use Tchebychev iterations when  $Y^{j_0}$  is already close to target values otherwise this method will diverge.

### 3.2 Coarse Level Correction: Z Method

The Z method of coarse level corrections for the nonlinear model is given as follows:

$$Y = Y^K + Q_0 E_{n'}^n Z', \quad (50)$$

here  $Y^K$  is the value obtained on the fine level,  $Z'$  is the coarse level correction we are looking for,  $Q_0 = \Omega_n P_n \Omega_n^T$ , and  $P_n$  is the same permutation matrix as we used for linear problem. The idea of the Z Method is to reverse pairing between eigenvalues and eigenvectors by multiplying the matrix  $Q_0$  so that larger eigenvalues pair with higher frequency on the coarse level and relaxations can remove high frequent errors efficiently [6].

By using coarse level correction (50) in the original nonlinear problem (42), we can easily obtain the following relations for the  $Z'$  correction by chain rules:

$$\mathcal{J}'(Z') = E_{n'}^{nT} \cdot Q_0 \cdot \mathcal{J}'(Y) = A'_{\mathcal{Z}'} Z' - b'_{\mathcal{Z}'}, \quad (51)$$

here  $A'_{\mathcal{Z}'}$  stands for the coarse level Jacobian matrix of the Z method (we share the same notation with Y method for simplicity), its value can be obtained by

$$A'_{\mathcal{Z}'} = E_{n'}^{nT} Q_0 A_{\mathcal{J}'} Q_0 E_{n'}^n, \quad (52)$$

and the constant vector  $b'_{\mathcal{Z}'}$

$$b'_{\mathcal{Z}'} = E_{n'}^{nT} Q_0 (b_{\mathcal{J}'} - A_{\mathcal{J}'} Y^K). \quad (53)$$

Thus we can again employ classical steepest-descent iterations on the coarse level for correction by initializing  $Z'_0 = 0$ :

$$Z' \leftarrow Z' - \rho \mathcal{J}'(Z'), \quad (54)$$

then we can update by  $Y^K + Q_0 E_{n'}^n Z'$  on the fine level. Iterations on the fine level (44) and corrections on the coarse level (54) complete another two-level correction-type ideal algorithm for the nonlinear model problem. To speed up the rate of convergence, we can use Tchebychev iterations [7] on the fine level.

### 3.3 Coarse Level Correction: $L$ Method

$L$  Method uses the idea of the  $Z$  Method to reverse pairing between eigenvalues and eigenvectors by a simple matrix  $\Delta$ , given the previous section, instead of computing the complicated matrix  $Q_0$ . Thus the two methods are almost similar except replacing matrix  $Q_0$  by matrix  $\Delta$  in the  $Z$  Method, i.e., the  $L$  method of coarse level corrections:

$$Y = Y^K + \Delta E_{n'}^n L', \quad (55)$$

here  $Y^K$  is the value obtained on the fine level,  $L'$  is the coarse level correction we are looking for.

By using coarse level correction (50) in the original nonlinear problem (42), we can easily obtain the following relations for the  $L'$  correction by:

$$\mathcal{J}'(L') = E_{n'}^n{}^T \cdot \Delta \cdot \mathcal{J}'(Y) = A'_{\mathcal{J}'} L' - b'_{\mathcal{J}'}, \quad (56)$$

here  $A'_{\mathcal{J}'}$  stands for the coarse level Jacobian matrix of the  $L'$  method, its value can be obtained by

$$A'_{\mathcal{J}'} = E_{n'}^n{}^T \Delta A_{\mathcal{J}'} \Delta E_{n'}^n, \quad (57)$$

and the constant vector  $b'_{\mathcal{J}'}$

$$b'_{\mathcal{J}'} = E_{n'}^n{}^T \Delta (b_{\mathcal{J}'} - A_{\mathcal{J}'} Y^K). \quad (58)$$

Thus we can again employ the classical steepest-descent iterations on the coarse level for correction by initializing  $L'_0 = 0$ :

$$L' \leftarrow L' - \rho \mathcal{J}'(L'), \quad (59)$$

then we can update by  $Y^K + \Delta E_{n'}^n L'$  on the fine level. Iterations on the fine level (44) and corrections on the coarse level (59) complete another two-level correction-type ideal algorithm for the nonlinear model problem. To speed up the rate of convergence, we can use Tchebychev iterations [7] on the fine level.

### 3.4 Numerical Experiments

In all the experiments, we take the following initial guess and we elevate their degrees by using the degree elevation matrix  $E_{n'}^n$  ( $n = 8, n' = 4$ ):

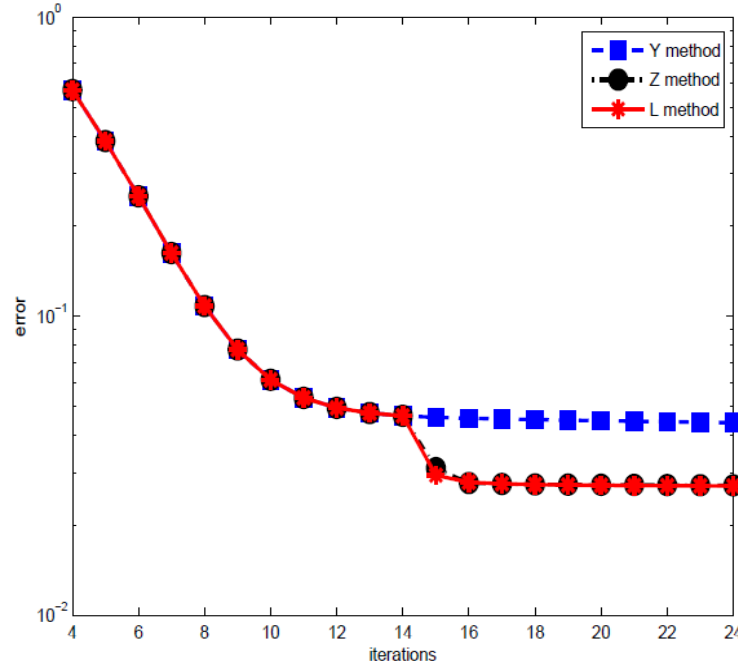
$X^0$	0	0	0.077	0.409	1.0
$Y^0$	0	0.01	0.01	0.01	0

To analyse our numerical experiences, we define the "error" as the difference between the approximation  $\mathcal{J}(Y)$  obtained by different methods concerned in our work and the exact solution.

For the first numerical experience, we employ 15 iterations on the fine level, and solve the problem completely on the coarse level for both  $Y$  and  $Z$  methods, then go back to the fine level to update values, finally do another 10 iterations. We plot errors versus the number of iterations in Fig. 3, we can observe that  $Y$  method is the most efficient in term of error reduction, and after corrections the  $Y$  method converges slowly (it still needs hundreds of iterations to fully converge), while  $Z$  and  $L$  methods get their full convergence for this case.

We define one two-level process as: first employ 35 iterations on the fine level, then do 5 iterations on the coarse level and update values on the fine level.  $\rho$ 's in these iterations on both levels are taken as multiplicative inverses of maximum eigenvalues of these iteration matrices. In our second experience, we repeat the two-level process three times and at the end of it do another 60 iterations on the fine level. We plot errors versus the number of iterations in Fig. 4. We can see that for coarse corrections of  $Y$  method does nothing good, while coarse corrections of  $Z$  and  $L$  methods are efficient. At the first beginning, it may appear contradictions with the linear model problem since errors obtained by  $L$  method are smaller than those obtained by  $Y$  method for this nonlinear model problem. Note that for the linear case we can only obtain the exact Jacobian matrix  $A_{\mathcal{J}'}$  and  $Q_0$  for  $Z$  method, which is not the case for nonlinear ones. Thus, it is no wonder that results obtained by the  $L$  method is the best among all considered methods for nonlinear model problems.

From the last experience, we can see that it takes too many iterations to achieve full convergence. To speed up the rate of convergence, we use Tchebychev iterations (HF) on the fine level ( $n = 8$ ) in our third experience. First we employ 5 iterations ( $\rho$ 's are chosen as multiplicative inverses of maximum eigenvalues of Jacobian matrices) on the fine level (so that solutions are close to exact ones in some sense, which makes sure that Tchebychev iterations converge, since initial values are very bad), then we can employ 6 Tchebychev iterations (HF) on the fine level, and solve them on coarse level ( $n' = 4$ ) "exactly", finally go back to the fine level and do another 6 Tchebychev iterations. We put errors corresponding to the number of iterations in Fig. 5. It is obvious that  $Z'$  and  $L'$  methods converge much faster, while corrections of  $Y$  method doesn't make any difference.



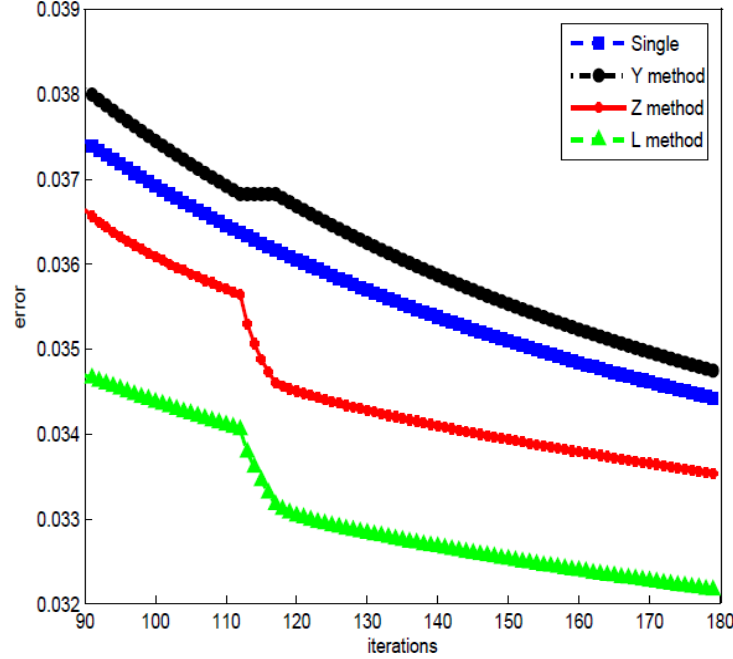
**Figure 3.** Comparison between the results for  $Y$ ,  $Z$  and  $L$  methods for the nonlinear model problem; First experiment: we employ 15 iterations on the fine level ( $n = 8$ ), and solve completely on coarse level, then go back the fine level to update values, finally do another 10 iterations

Through our numerical experiences, we find some other interesting results:

1. In two-level ideal algorithms, if we compute the same number of iterations on the fine level ( $n = 8$ ) and solve "exactly" on the coarse level, the results are better for the coarse level corresponding to  $n' = 4$  than  $n' = 2$ . As far as the cost of operations is concerned, the case  $n' = 2$  uses less operations on the coarse level than  $n' = 4$ . Thus, there is not an obvious winner for this nonlinear problem.
2. In three-level ideal algorithms of  $Z$  and  $L$  methods ( $n'' = 2$ ,  $n' = 4$ , and  $n = 8$ ), we don't need to employ the transformation matrices  $Q_0$  or  $\Delta$  on the coarse level to repair eigenvalues and eigenvectors.
3. On the bottom of  $\mathcal{V}$  cycle, we can also use Newton iterations to solve for coarse corrections instead of solving them "exactly".
4. In  $Z'$  method, we find that the algorithms converges faster when we use the Jacobian matrix of the linear model problem to compute the matrix  $Q_0$ .

## 4. Conclusion

In this paper, a number of variants of two-level correction algorithm has been presented for a linear and nonlinear problem. We have demonstrated that the method ( $Z$  and  $L$ ) based on the classical modal-analysis concept is the



**Figure 4.** Comparison between the results for Y, Z and L methods for the nonlinear model problem; Second experiment: we employ 35 iterations on the fine level, then do 5 iterations on the coarse level and update values on the fine level.  $\rho$ 's in these iterations on both levels are taken as multiplicative inverses of maximum eigenvalues of these iteration matrices.

most efficient. For the both cases, we use a model problem which consists to minimize a quadratic functional using the steepest descent algorithms; the optimizer could be viewed as the classical point-Jacobi iteration applied to a certain matrix.

It is well known that there are two major multigrid methods [11][12] to deal with nonlinear problems: Newton-multigrid and Full Approximation Scheme (FAS). The algorithm used in this paper could be viewed as the Newton-multigrid. As perspective, we propose applying the Full Approximation Scheme for the nonlinear model problem  $A(Y) = f$  as follow:

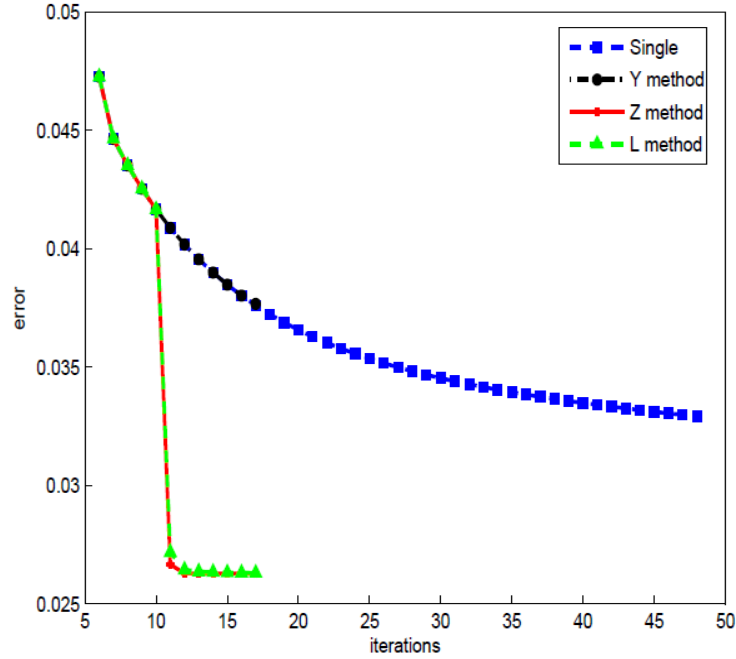
1. on the fine level  $n$ , relax the nonlinear problem  $AY^j = f$  by initializing  $Y^0$ , and use  $Y^K$  as the values on the fine level after relaxations;
2. on the coarse level correction, we need to solve the following residual equation:

$$A'(Y') = A'(R_n^{n'} Y) + R_n^{n'} (f - A(Y^K)), \quad (60)$$

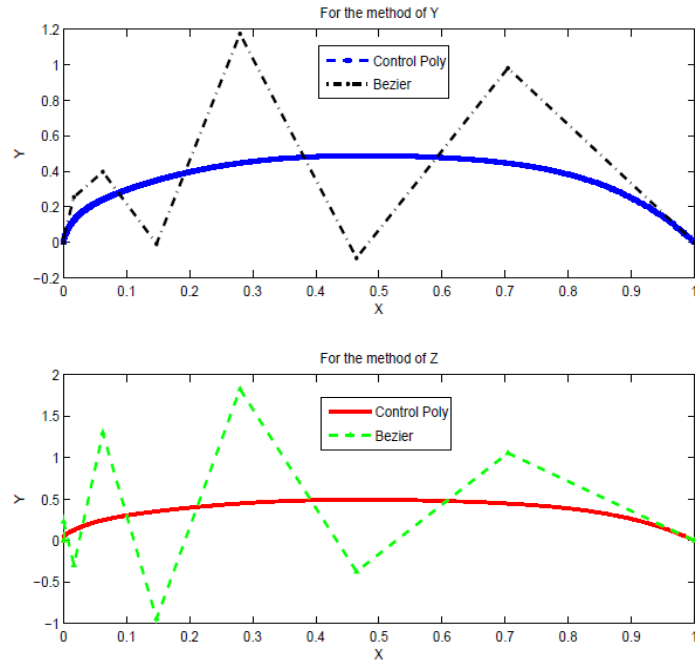
where  $Y'$  is the approximation on the coarse level, that is the reason how the name of this algorithm FAS comes from,  $A'$  stands for the nonlinear operator on the coarse level  $n'$  and  $R_n^{n'}$  is used here as the restriction operator which projects from the fine level  $n$  to the coarse level  $n'$ ;

3. back to the fine level  $n$ , update the values  $Y^K$  by  $Y^K + E_{n'}^n (Y' - R_n^{n'} Y^K)$ , here  $E_{n'}^n$  is the degree elevation operator.

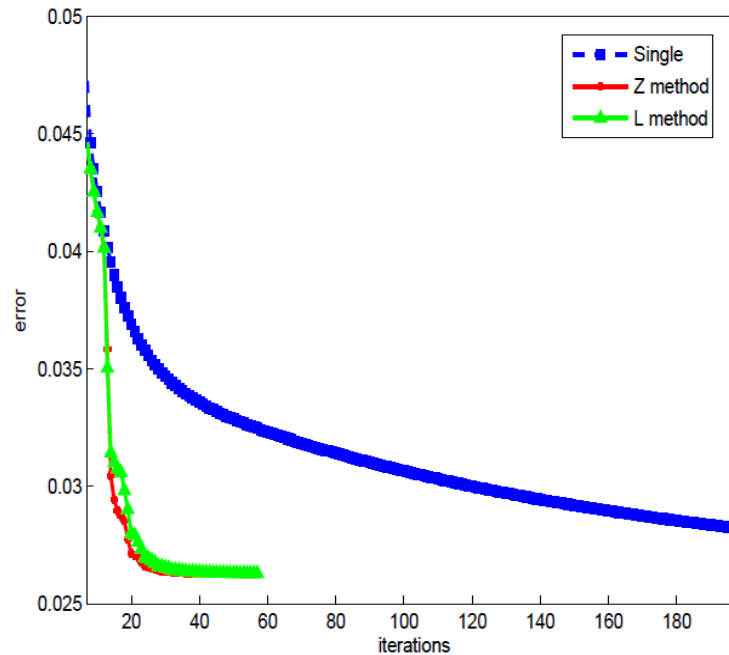
In the two level ideal algorithm of our nonlinear model problem, the linearized approach is identical with the algorithm of FAS when the coarse level is solved "exactly".



**Figure 5.** Comparison between the results for Y, Z and L methods for the nonlinear model problem; Third experiment: we employ 5 iterations on the fine level, then we can employ 6 Tchebychev iterations (HF) on the fine level, and solve on the coarse level "exactly", finally go back to the fine level and do another 6 Tchebychev iterations.



**Figure 6.** Control polygon and Bézier profile for Y and Z methods.



**Figure 7.** Comparaision between Z and L methods for the nonlinear model problem; we use three levels (fine level  $n = 8$ , intermediate level  $n' = 4$ , and coarse level  $n'' = 2$ ).

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